EXERCISE – V | JEE PROBLEMS

1. (a) Evaluate $\int_0^{\pi/2} \frac{\cos^9 x}{\cos^3 x + \sin^3 x} dx$ **[REE 2001, 3+5]**

(b) Evaluate $\int_0^{\pi} \frac{x dx}{1 + \cos\alpha \sin x}$

2. (a) Let $f(x) = \int_1^x \sqrt{2 - t^2}\, dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are **[JEE 2002(Scr.), 3+3+3]**

(A) ± 1 (B) $\pm \frac{1}{\sqrt{2}}$ (C) $\pm \frac{1}{2}$ (D) 0 and 1

(b) Let T > 0 be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x+T) = f(x)$.

If $I = \int_0^T f(x)\, dx$ then the value of $\int_3^{3+3T} f(2x)\, dx$ is

(A) $\frac{3}{2} I$ (B) 2 I (C) 3 I (D) 6 I

(c) The integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left([x] + \ln\left(\frac{1+x}{1-x}\right) \right) dx$ equals

(where [∗] denotes greatest integer function)

(A) $-\frac{1}{2}$ (B) 0 (C) 1 (D) $2\ \ell n \left(\frac{1}{2}\right)$

3. If f is an even function then prove that

$$\int_0^{\pi/2} f(\cos 2x) \cos x\, dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x\, dx$$

[JEE 2003 (Mains), 2]

4. (a) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx =$ **[JEE 2001]**

(A) $\frac{\pi}{2} + 1$ (B) $\frac{\pi}{2} - 1$ (C) π (D) 1

(b) If $\int_0^{t^2} x f(x)\, dx = \frac{2}{5} t^5$, $t > 0$, then $f\left(\frac{4}{25}\right) =$

[JEE 2004, (Scr.)]

(A) 2/5 (B) 5/2 (C) –2/5 (D) 1

(c) If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x . \cos\sqrt{\theta}}{1 + \sin^2\sqrt{\theta}} d\theta$ then find $\frac{dy}{dx}$ at $x = \pi$.

[JEE 2004 (Mains), 2]

(d) Evaluate $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$ **[JEE 2004 (Mains), 4]**

5. (a) If $\int_{\sin x}^{1} t^2 (f(t))\, dt = (1 - \sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

[JEE 2005 (Scr.)]

(A) 1/3 (B) $1/\sqrt{3}$ (C) 3 (D) $\sqrt{3}$

(b) $\int_{-2}^{0} \left(x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)\right) dx$ is equal to

[JEE 2005 (Scr.)]

(A) –4 (B) 0 (C) 4 (D) 6

(c) Evaluate : $\int_0^{\pi} e^{|\cos x|} \left(2 \sin\left(\frac{1}{2}\cos x\right) + 3 \cos\left(\frac{1}{2}\cos x\right) \right) \sin x\, dx$.

[JEE 2005, Mains, 2]

6. Let y = f(x) be a twice differentiable, non-negaitve function defined on [a, b]. The area $\int_a^b f(x) dx$, b > a bounded by y = f(x), the x-axis and the ordinates at x = a and x = b can be approximated as

$$\int_a^b f(x) dx \cong \frac{(b-a)}{2} \{f(a) + f(b)\}.$$

Since $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, $c \in (a, b)$, a better approximation to $\int_a^b f(x) dx$ can be written as

$\int_a^b f(x)dx \cong \frac{(c-a)}{2}\{f(a)+f(c)\} + \frac{(b-c)}{2}\{f(c)+f(b)\} \equiv F(c)$.

If $c = \frac{a+b}{2}$, then this gives : **[JEE 2006]**

$\int_a^b f(x)dx \cong \frac{b-a}{4}\{f(a) + 2f(c) + f(b)\}$, ...(1)

(a) The approximate value of $\int_0^{\pi/2} \sin x\,dx$ using rule (1) given above is

(A) $\frac{\pi}{8\sqrt{2}}(1 + \sqrt{2})$ (B) $\frac{\pi}{4\sqrt{2}}(1 + \sqrt{2})$

(C) $\frac{\pi}{8}(1 + \sqrt{2})$ (D) $\frac{\pi}{4}(1 + \sqrt{2})$

(b) If $\lim_{t \to a} \left\{ \frac{\int_0^t f(x)dx - \frac{(t-a)}{2}(f(t)+f(a))}{(t-a)^3} \right\} = 0$, for each fixed a, then f(x) is a polynomial of degree utmost

(A) 4 (B) 3 (C) 2 (D) 1

(c) If f ''(x) < 0, x ∈ (a, b), then at the point C (c, f(c)) on y = f(x) for which F(c) is a maximum, f '(c) is given by

(A) $f'(c) = \frac{f(b)-f(a)}{b-a}$ (B) $f'(c) = \frac{f(b)-f(a)}{a-b}$

(C) $f'(c) = \frac{2(f(b)-f(a))}{b-a}$ (D) $f'(c) = 0$

7. Find the value of $\frac{5050\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ **[JEE 2006, 6]**

8. (a) $\lim_{x \to \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ equals **[JEE 2007, 3 + 6]**

(A) $\frac{8}{\pi}f(2)$ (B) $\frac{2}{\pi}f(2)$ (C) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (D) 4f(2)

(b) Match the integrals in **Column I** with the values in **Column II**.

Column I		Column II	
(A)	$\int_{-1}^{1}\frac{dx}{1+x^2}$	(P)	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B)	$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(Q)	$2\log\left(\frac{2}{3}\right)$
(C)	$\int_2^3 \frac{dx}{1-x^2}$	(R)	$\frac{\pi}{3}$
(D)	$\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(S)	$\frac{\pi}{2}$

9. Let $S_n = \sum_{k=1}^{n}\frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=1}^{n-1}\frac{n}{n^2+kn+k^2}$, for n = 1, 2, 3, ... Then, **[JEE 2008, 4]**

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$

(C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

10. (a) Let f be a non-negative function defined on the interval [0,1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)dt$, $0 \le x \le 1$, and f(0) = 0, then **[JEE 2009, 3 + 4 + 4]**

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(b) If $I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx$, n = 0, 1, 2, ... then

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$ (D) $I_n = I_{n+1}$

(c) Let f : R → R be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of f(ℓn 5) is

11. (a) The value of $\lim\limits_{x\to 0}\frac{1}{x^3}\int\limits_0^x \frac{t\,\ell n(1+t)}{t^4+4}dt$ is **[JEE 2010, 3 + 3 + 5 + 3]**

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

(b) The value(s) of $\int\limits_0^1 \frac{x^4(1-x)^4}{1+x^2}$ dx is (are)

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

(c) Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x} f(x) = 2+\int\limits_0^x \sqrt{t^4+1}\,dt$, for all x ∈ (–1, 1), and let f^{-1} be the inverse function of f. Then $(f^{-1})'$ (2) is equal to

(A) 1 (B) 1/3 (C) 1/2 (D) 1/e

(d) For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$$

Then the value of $\frac{\pi^2}{10}\int\limits_{-10}^{10} f(x)\cos \pi x$ dx is

12. The value of $\int\limits_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2+\sin(\ln 6-x^2)}$ dx is **[JEE 2011, 4]**

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

13. Let f : [1, ∞) → [2, ∞) be a differentiable function such that f(1) = 2. If $6\int\limits_1^x f(t)\,dt = 3x\,f(x)-x^3$ for all x ≥ 1, then the value of f(2) is **[JEE 2011, 4]**

14. The value of the integral $\int\limits_{-\pi/2}^{\pi/2}\left(x^2+\ell n\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$